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NONLINEAR EFFECTS UPON WAVES NEAR CAUSTICS

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CONTENTS

According to linear theory the wave intensity of a slowly varying wave train becomes particularly large near caustics. In this paper it is shown how the waves are modified when the wave intensity is sufficient for nonlinear effects to begin to be important. Two types of near-linear caustics can arise in which nonlinearity either tends to advance or to retard the reflexion of waves from the caustic. General examples are given in terms of one-dimensional wave propagation, and of propagation in a uniform medium. Detailed consideration is given to a particular example: small-amplitude water waves on deep currents. This helps to provide an interpretative framework for the large-amplitude results presented in the companion paper (Peregrine & Thomas 1979). For the more exceptional case of triple roots, or cusped caustics, the increase in wave intensity is even more dramatic. In three appendices the analysis for caustics is extended to some higher-order cases.

1. Introduction

In many wave fields the most prominent features are caustics and their singularities. Ray theory for infinitesimal slowly varying waves predicts infinite wave intensities at these envelopes of the group-velocity paths (or rays). It is the exceptionally large, but finite, wave intensity that makes such features so noticeable and so important. There are two properties of the singularity which indicate that the simple ray solution is invalid. The wave amplitude is not slowly varying, and the waves are not infinitesimal.

Linear analyses of the wave field near caustics date back to Airy (1838). The rapid variation of the wave amplitude is allowed for by the retention of extra terms. These higher-order dispersive (h.o.d.) terms give rise to solutions involving Airy functions (Miller 1946; Abramowitz & Stegun 1964, § 10.4). Recent examples of such work are by Ludwig (1966) on uniform expansion, Smith (1970) on whispering-gallery waves, and Richards (1973) on seismic waves. Linear theory gives a solution with no singularities, but this may be inadequate if an approximation of infinitesimal amplitude has been made. Present interests in linear waves tend to be directed towards higher-order generic 'singularities' of which the simplest is a cusp of a caustic (see, for example, Berry 1976, Budden 1976).

Nonlinear analyses of waves near caustics have largely concerned non-dispersive waves; for example, Ostrovosky (1976) gives a review of results for acoustic and shallow-water waves. Bobbitt & Cumberbatch (1976) also derive equations for shallow-water waves in water of varying depth. A distinctive feature of the results for non-dispersive waves is that the wave envelopes are necessarily unsteady.

This paper presents results for weakly-nonlinear dispersive waves near caustics. It is found that there are steady solutions for the wave envelope in which nonlinearity is balanced by a change of wavenumber. The major result is that caustics fall into two distinct categories, these are given the labels R and S. Examples of each category are given for deep water waves on currents. Much of this work arose from attempts to relate and interpret the results of Smith (1976*a*) and of Peregrine & Thomas (1979), which is referred to as P. & T. in the following, on different facets of this particular physical example.

The basic results of general slowly varying wave theory are presented in § 2. An averaged Lagrangian is used (Whitham 1974), since averaging of equations of motion gives equivalent results. That is, for linear waves propagation can be described in terms of the wave action following paths (rays) at the local group-velocity. This interpretation does not extend to near-linear waves and a fuller discussion of group velocity in this context is given in § 6 of the companion paper P. & T.

Caustics are introduced in § 3. They arise as envelopes of rays in the linear theory which lead to a singularity of the amplitude where the approximation becomes invalid. The simplest near-linear theory shows two very different types of solution in the neighbourhood of a linear caustic. The R solution has a singularity at finite amplitude before the linear caustic position is reached, and the S solution penetrates beyond the linear caustic position and exhibits a rapid growth in amplitude.

In order to illustrate how the two types of caustic can arise two classes of wave problems are investigated. Section 3 gives an analysis of wave propagation in a non-uniform medium for wave and medium properties varying along a single direction. Section 4 gives a local analysis of three-dimensional wave propagation near a curved caustic in a uniform medium. In each class of problems both categories of caustic can arise.

Section 5 looks at the specific physical example of water waves on deep currents. It is determined upon which currents caustics can occur and to which category the caustics belong. Similar results can be expected to arise for other dispersive waves in a moving medium.

The singularities in ray solutions at caustics can be avoided by finding uniform solutions; these solutions involve Airy functions for linear equations. The linear case is examined in § 6 by using the operator expansion method which provides a particularly simple derivation from the dispersion relation. Since we have found that many users of ray solutions are either unaware of uniform caustic solutions, or find it difficult to use them, we also give explicit formulae for determining a uniform description of the waves from the non-uniform ray description.

Section 7 proceeds to use the same approach heuristically to examine the composite effect of h.o.d. and nonlinearity near caustics. The heuristic results agree with those derived by Smith (1976*a*) and for the two categories of caustic the role of the Airy function in linear theory is taken over by either of two Painlevé transcendents. Recent work by Kaup & Newell (1978) leads to some solutions of the corresponding unsteady equations which are discussed. The results of this section are qualified by the information given in P. & T. and in the conclusion an analogy is drawn between wave behaviour near an S caustic and waves on a beach. Observations of water waves suggests that in practice R caustics are usually regular and that S caustics are frequently singular and involve wave breaking. Of course, this assumes that the waves are capable of being described by near-linear theory as they approach the caustics and that wave amplitudes have not become too great in the approach to the caustic.

The analysis for caustics provides a basis for the study of higher-order ray singularities. The appendices all concern the case in which there is a three-fold coalescence of adjacent rays. In appendix A it is shown that for one-dimensional propagation three types of near-linear triple roots can arise, characterized respectively in terms of reflexion, strong amplification, and transmission across the triple-root position. The comments made earlier concerning the practical use of uniform solutions apply with even more force for cusped caustics, or triple-roots. Appendix B provides general formulae which permit the uniform linear solution to be constructed from the readily calculable, but highly singular, ray solutions near a cusp of a caustic. Finally, appendix C gives a derivation of the nonlinear partial differential equation which governs the wave height near a triple-root when both nonlinear and high-order dispersive effects are important.

2. Slowly varying near-linear waves

In all this work we assume that all length and time scales are very much greater than the wavelength and period of the waves. It is possible to introduce a small parameter corresponding to the smallness of the wave scale, but, since all the results in this paper concern different 'first approximations' in this parameter, formal perturbation expansions are not necessary.

The other small parameter in this problem is a measure of the wave amplitude. This is carried beyond the first approximation but again a perturbation expansion would introduce unnecessary formalism since throughout the paper we assume that the appropriate plane wave solution and first correction to the dispersion equation are known. That is, we consider waves such that in a homogeneous medium, and when the wave amplitude is small, there is a plane-wave solution of the form

$$u(x, y, t) = u_1 + u_2 + \dots . \tag{2.1}$$

Here the leading order term u_1 has the form

$$u_1 = \mathrm{Re}\,[af(\boldsymbol{y})\,\mathrm{e}^{\mathrm{i}\chi}], \quad \chi = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t \tag{2.2, 3}$$

and the solution is near-linear in the sense that

$$u_n(\boldsymbol{y}, \chi) = O(a^n). \tag{2.4}$$

It is of course implicit that the solution should be a sensible approximation to the physical waves it models. The wave function u, and $f(\boldsymbol{y})$, may be a scalar or a vector (e.g. a velocity potential, or a velocity field). A point in the propagation space is defined by the vector $\boldsymbol{x}$ and in the transverse (or modal) space by $\boldsymbol{y}$. For example, for surface gravity waves, $\boldsymbol{x}$ is a two-dimensional horizontal position vector, and $\boldsymbol{y}$ is a one-dimensional vertical coordinate (for more details see Hayes 1970).

If a Lagrangian exists, an averaged Lagrangian may be found by substituting the plane-wave solution (2.1) and averaging with respect to χ. Typically the averaged Lagrangian has the form

$$\mathscr{L} = G(\boldsymbol{x}, t, \omega, \boldsymbol{k})\,a^2 + \tfrac{1}{2}H(\boldsymbol{k}, \boldsymbol{x}, t)\,a^4, \tag{2.5}$$

where G, H, ω, $\boldsymbol{k}$ and a are all assumed to be slowly-varying functions of $(\boldsymbol{x}, t)$, and a is written in place of $|a|$ for simplicity. Only the first nonlinear effects on wave propagation have been included and any dependence of H on ω has been eliminated by using the linear dispersion relation

$$G(\omega, \mathbf{k}) = 0. \tag{2.6}$$

The above representation (2.5) does not include all cases where a potential is used, such as for water waves on finite-water depth. For more details see Whitham (1974). Caustics have not been studied in this case. Note for comparison with some other work that if instead of assuming the form (2.2), the form

$$u_1 = a_1 f(\boldsymbol{y})\,\mathrm{e}^{\mathrm{i}\chi} + \text{complex conjugate} \tag{2.7}$$

were taken, then

$$\mathscr{L} = 4Ga_1^2 + 8Ha_1^4. \tag{2.8}$$

Consistency relations between ω and $\boldsymbol{k}$ are obtained by assuming they arise thus,

$$\omega = -\partial\chi/\partial t, \quad \boldsymbol{k} = \partial\chi/\partial\boldsymbol{x}, \tag{2.9}$$

from a phase function χ so that consideration of the second derivatives of χ yields

$$\frac{\partial \boldsymbol{k}}{\partial t} = \frac{\partial \omega}{\partial \boldsymbol{x}}, \quad \frac{\partial}{\partial \boldsymbol{x}} \wedge \boldsymbol{k} = 0. \tag{2.10, 11}$$

We use $\partial/\partial\boldsymbol{x}$ to denote the vector operator $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ and distinguish between partial derivatives in the space $(\boldsymbol{x}, t)$ and partial derivatives in the space $(\omega, \boldsymbol{k}, \boldsymbol{x}, t)$ by using subscripts for the latter. For example

$$\frac{\partial G}{\partial t} = G_\omega \frac{\partial \omega}{\partial t} + G_{\boldsymbol{k}}\cdot\frac{\partial \boldsymbol{k}}{\partial t} + G_t. \tag{2.12}$$

The Euler–Lagrange equations of the variational principle

$$\delta\mathscr{L} = 0$$

for variations of a and χ are

$$\mathscr{L}_a = 0 \quad \text{and} \quad \frac{\partial \omega}{\partial t} - \frac{\partial}{\partial \boldsymbol{x}}\cdot\mathscr{L}_{\boldsymbol{k}} = 0. \tag{2.13, 14}$$

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On using the expression (2.5) for $\mathscr{L}$, these become

$$G + Ha^2 = 0, \tag{2.15}$$

the dispersion relation, and

$$\frac{\partial}{\partial t}(G_\omega a^2) - \frac{\partial}{\partial \boldsymbol{x}} \cdot (G_{\boldsymbol{k}} a^2 + \tfrac{1}{2} H_{\boldsymbol{k}} a^4) = 0, \tag{2.16}$$

the equation for conservation of wave-action, $G_\omega a^2$.

The linearized system of equations (2.10, 11, 15 and 16) obtained by putting $H = 0$, have a single set of characteristic directions given by

$$\partial \boldsymbol{x}/\partial t = -G_{\boldsymbol{k}}/G_\omega = \omega_{\boldsymbol{k}} = \boldsymbol{c}_{\mathrm{g}}, \tag{2.17}$$

where $\boldsymbol{c}_{\mathrm{g}}$ is the group velocity of the linear waves. Lines defined by (2.17) are often called rays. Solutions of the linear equations may be found by integrating in $(\boldsymbol{x}, t)$ along the rays from an initial surface (or line) with given initial values ω, $\boldsymbol{k}$ and a. The differentiated forms

$$\partial G/\partial t = 0, \quad \partial G/\partial \boldsymbol{x} = 0, \tag{2.18, 19}$$

of the dispersion relation, $G = 0$, are used to find ω and $\boldsymbol{k}$, and then amplitude is found from the wave-action equation.

The near-linear equations are more difficult to integrate since they either have distinct characteristics or complex characteristics. This also means that their solutions are often not uniformly valid, either because characteristics of the same family meet, or the solutions are unstable (see Whitham 1974, Lighthill 1967, Hayes 1973, P. & T.). We now consider how solutions to these equations behave in a region where these linear solutions become invalid.

3. Straight caustics

In many examples of wave propagation rays calculated from linear theory meet. In general if the waves are propagating in three (two, one) dimensions they meet on a surface (line, point) which is an envelope of rays and is called a caustic. The caustic may occur because of non-uniformities in the initial conditions, or in the propagating region or in both. The position of caustics may vary with time and will generally have some lines (points) at which the surface (line) has a singularity. For example, a cusp is the simplest generic singularity. In the main body of the present paper we do not attempt to deal with the more complex singularities, but some results for cusps are given in the appendices.

In this section we consider part of a caustic surface (line) which is locally plane (straight). Such a caustic can only occur in an inhomogeneous medium. Curved caustics in a uniform medium are discussed in the next section and it is clear how the two cases may be combined to cover curved caustics in a non-uniform medium. Attention is confined to the case of a steady caustic in a steady wave field. With these restrictions solutions relevant to the neighbourhood of a caustic may be found, since we may assume that variations of wave properties along the caustic are negligible compared to variations perpendicular to the caustic.

Let x be a coordinate measured perpendicular to the caustic with $x = 0$ being the plane (line, point) of the caustic, let $\boldsymbol{i}$ be a unit vector in the $+x$ direction and let the waves approach the caustic from the $-x$ direction. As indicated above, we now suppose all wave properties to depend on x alone. The consistency relations (2.10, 11) immediately show that ω and $\boldsymbol{k} \wedge \boldsymbol{i}$

are constant and thus the amplitude, a, and the wave number component perpendicular to the caustic, $\boldsymbol{k}\cdot\boldsymbol{i} = l$ say, are the only dependent variables to be found. The values of the constants and boundary conditions for the variables are found by matching the caustic solution with the wavefield outside the caustic region.

In the case of linear waves, a and l are determined from the dispersion relation which might now be written as

$$G(x, l) = 0, \tag{3.1}$$

and from the wave action equation which integrates to

$$-G_l a^2 = \pm B, \tag{3.2}$$

where B is the constant incident wave-action flux. If we follow the convention that G_ω be positive, then the plus and minus signs refer to the incident and reflected waves respectively. The form of G must be such as to have solutions corresponding to both the incident and reflected waves; for example,

$$G(\boldsymbol{x}, \boldsymbol{k}, \omega) = \omega^2 - c^2(x)\, k^2$$

for a non-dispersive medium.

The caustic surface is an envelope of the rays; therefore, rays are tangential to it and thus the $\boldsymbol{i}$ component of the ray equation (2.17), that is

$$\mathrm{d}x/\mathrm{d}t = -G_l/G_\omega, \tag{3.3}$$

must be such that $\mathrm{d}x/\mathrm{d}t = 0$ at the caustic $x = 0$, since waves do not propagate past that point (line, plane) in this approximation. Hence, $G_l = 0$ and, from equation (3.2), the amplitude, a, is singular.

A mathematically convenient feature of the linear equations is that we can first solve the dispersion relation (3.1) for $l(x)$, and subsequently solve the wave action equation (3.2) for $a(x)$. In particular, it is straightforward to clarify the nature of the singularity.

At the caustic, $x = 0$, the dispersion relation defines the wave-number component l_0:

$$G(0, l_0) = 0. \tag{3.4}$$

If G has a Taylor series expansion then

$$G(x, l) = xG_x(0, l_0) + \tfrac{1}{2}(l - l_0)^2\, G_{ll}(0, l_0) + \dots; \tag{3.5}$$

hence

$$l \simeq l_0 \pm (2G_x/G_{ll})^{\frac{1}{2}}\, (-x)^{\frac{1}{2}}, \tag{3.6}$$

and, from equation (3.2)

$$a \simeq (B^2/2G_x/G_{ll})^{\frac{1}{4}}\, (-x)^{-\frac{1}{4}}. \tag{3.7}$$

For the near-linear equations we cannot proceed in the same way. Rays do not necessarily exist and there is not an immediate criterion to determine where the wave amplitude is greatest. It also proves to be of little value to look for singularities in the solutions. The best approach is to realise that, for solutions to be realistic, amplitudes must be small and thus solutions should depart little from the linear solution for infinitesimal amplitude.

Consider the (x, l) plane. The linear solution is a line given by the dispersion relation (3.1) and at a caustic its tangent is parallel to the l axis since $G_l = 0$. For examples see figures 6 and 7. The near-linear dispersion relation is

$$G + Ha^2 = 0, \tag{3.8}$$

and thus as long as H is non-zero the solution curves for small values of a must lie on one side of the curve $G = 0$. We shall not consider special cases where H may be zero near a caustic.

The near-linear wave-action equation is

$$-(G_l+\tfrac{1}{2}H_l a^2)a^2 = \pm B. \tag{3.9}$$

Now, a^2 may be eliminated between (3.8) and (3.9) to give

$$G(HG_l-\tfrac{1}{2}GH_l)/H^2 = \pm B. \tag{3.10}$$

For constant B this defines solution curves in the (x, l) plane. However, B must be small, $O(a^2)$, so that the solution curves are close to the lines given by $B = 0$. That is the lines

$$G = 0 \quad \text{and} \quad HG_l - \tfrac{1}{2}GH_l = 0. \tag{3.11, 12}$$

These two lines clearly intersect at the linear caustic point

$$G = 0, \quad G_l = 0. \tag{3.13}$$

Once the two lines (3.11, 12) are found it is easy to sketch solutions for small B, since the sign of H in the dispersion relation (3.4) indicates which side of the linear solution, $G = 0$, the solution line lies.

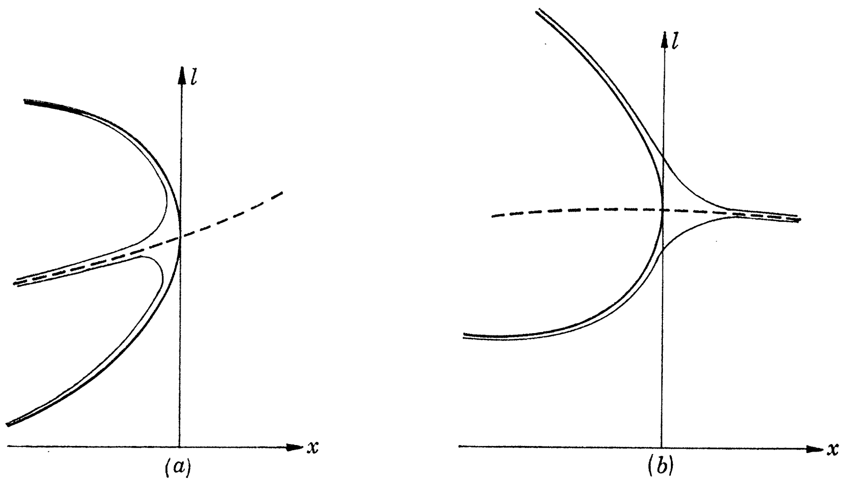

FIGURE 1. Diagrams illustrating the behaviour of near-linear waves close to a linear caustic point. Heavy lines represent linear solutions. Dashed lines represent $HG_l - \frac{1}{2}GH_l = 0$. Thin continuous lines represent solutions of simple slowly-varying near-linear theory. (*a*) Type R caustic; (*b*) type S caustic.

Solution lines may take one of two different forms, depending on the sign of H/G_x at the linear caustic point. We call them type R and type S for H/G_x positive and negative respectively. The two types of curve are sketched in figure 1. (Note that G_x and G_{ll} have the same sign, by our choice of x, so H/G_{ll} is also a distinguishing parameter.)

For a type R caustic the solution lines are inside the curve (strictly on the concave side of the curve) of the linear solution. There are two branches of the nonlinear solution which correspond to incident and reflected waves. Unlike the linear solution these do not meet at a caustic point. However, each branch does have a singularity where its tangent is perpendicular to the x axis. This singularity implies a rapid variation of wave number with x and hence this approximation becomes invalid before the singularity is reached. It is of particular interest to note that the singularity is at a sufficiently small amplitude that the near-linear approximation can be expected to remain valid. It occurs *before* the waves reach the linear caustic point.

For a type S caustic the situation is quite different. As the linear solution approaches its singularity so the near-linear solution diverges from it to follow the second of the lines (3.11).

There is no singularity for the wave amplitude. However, as the solution diverges from the linear solution so the amplitude of the waves increases and the near-linear approximation becomes inappropriate. The subsequent behaviour of the waves then depends on their finite-amplitude properties. For example, in the case examined in § 5 of P. & T. the finite-amplitude solution is qualitatively similar to a near-linear solution right up to the steepest possible wave train.

For both R and S caustics uniform linear solutions exist as shown in § 6, but the amplitude of those solutions can be too large for linear theory to be valid.

The properties of solutions near these singularities may be found by Taylor series expansion of G, G_l and H about the linear caustic point. The leading terms in the dispersion equation and wave-action equation may be written

$$xG_x + \tfrac{1}{2}(l-l_0)^2\, G_{ll} + a^2 H = 0, \tag{3.14}$$

and

$$xG_{xl} + (l-l_0)\, G_{ll} + \tfrac{1}{2}a^2 H_l = -B/a^2, \tag{3.15}$$

where the derivatives are all evaluated at $(0, l_0)$. In the dispersion equation, the first two terms balance for linear waves, so it may be expected that all three terms are of the same order for near-linear waves. If that is so then the leading terms of the wave-action equation (3.15) are

$$(l-l_0)\, G_{ll} = -B/a^2, \tag{3.16}$$

and hence, on eliminating $(l-l_0)$ from (3.14),

$$xG_x + \tfrac{1}{2}B^2/G_{ll}a^4 + a^2 H = 0. \tag{3.17}$$

A measure of the energy amplification is given by

$$A = a^2/|B|, \tag{3.18}$$

and with the introduction of the quantities

$$X = 2xG_x G_{ll} \quad \text{and} \quad b = 2G_{ll}H|B|, \tag{3.19, 20}$$

equation (3.17) takes the simple form

$$X + 1/A^2 + bA = 0. \tag{3.21}$$

The nonlinearity of the waves is proportional to b. The solution is only singular for $b \geqslant 0$, in which case the singularity occurs at

$$A^3 = 2/b, \quad X = -3(\tfrac{1}{2}b)^{\frac{2}{3}}. \tag{3.22}$$

The behaviour of solutions is shown in figure 2.

However, this is not the only possibility; for small, or zero, values of B all the terms on the left hand side of equation (3.15) may be of the same order. In that case the leading terms in the dispersion equation are

$$xG_x + a^2 H = 0, \tag{3.23}$$

or

$$X + bA = 0. \tag{3.24}$$

However it may be seen in figure 2 that these correspond to the solution curves for type R caustics, $b \geqslant 0$, after the singularity is met. For type S caustics, $b < 0$, this solution is more relevant. Note, it is not *just* an asymptote of the solution (3.21) since the variation of $(l-l_0)$ with x differs in these two limiting cases. In the first case by defining

$$K = (l-l_0)\,G_{ll} \tag{3.25}$$

we obtain
$$X+K^2-b\,\mathrm{sgn}\,(B)/K = 0, \tag{3.26}$$

whereas in the second case
$$K+hX+b\,\mathrm{sgn}\,(B)/X = 0, \tag{3.27}$$

where
$$h = (2G_{xl}H+G_xH_l)/(4G_xG_{ll}H). \tag{3.28}$$

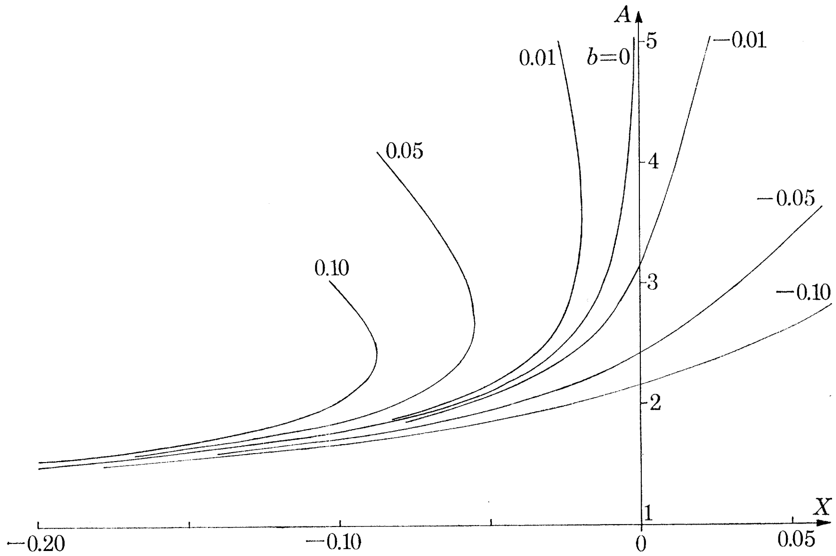

FIGURE 2. The amplitude of near-linear slowly varying waves close to a caustic: local solution. Type R caustics have $b > 0$. Type S caustics have $b < 0$. The line $b = 0$ is the linear solution, which meets $A = 1$ at $X = -1$. $A = a|B|^{-\frac{1}{2}}$, $X = 2xG_xG_{ll}$, $b = 2G_{ll}H|B|$.

As has been indicated equations (3.23) and (3.27) truncated to

$$K+hX = 0 \tag{3.29}$$

are appropriate for solutions with $B = 0$.

There are many situations, particularly with boundaries, where this solution is relevant. Particular examples are given in P. & T. The solution is also of interest since it is intermediate between waves approaching a caustic, $B > 0$, and waves reflected from a caustic, $B < 0$, and thus may be of value in constructing local or uniformly valid solutions.

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4. Curved caustics in a uniform medium

(a) *Linear theory*

In a uniform medium rays are straight lines, and to determine the linear ray solution it suffices to know the wave amplitude and ray direction on some surface $\mathscr{S}$. Suppose that the surface and the ray direction respectively have the parameteric representations $\boldsymbol{r}(\xi^1, \xi^2)$ and $\boldsymbol{t}(\xi^1, \xi^2)$ in terms of generalized coordinates ξ^1, ξ^2. Then the rays can be represented as

$$\boldsymbol{r}(\xi^1, \xi^2)+\lambda\boldsymbol{t}(\xi^1, \xi^2), \tag{4.1}$$

where λ is the distance along the ray from the reference position on $\mathscr{S}$.

The vector separation between points on adjacent rays is

$$(\boldsymbol{r}_1+\lambda \boldsymbol{t}_1)\, \mathrm{d}\xi^1+(\boldsymbol{r}_2+\lambda \boldsymbol{t}_2)\, \mathrm{d}\xi^2+\boldsymbol{t}\,\mathrm{d}\lambda,$$

where the subscripts denote partial derivatives with respect to ξ^j. A necessary condition for the rays to meet is that the three vectors $\boldsymbol{r}_1+\lambda \boldsymbol{t}_1$, $\boldsymbol{r}_2+\lambda \boldsymbol{t}_2$, $\boldsymbol{t}$ be co-planar:

$$[\lambda^2(\boldsymbol{t}_1 \wedge \boldsymbol{t}_2)+\lambda(\boldsymbol{t}_1 \wedge \boldsymbol{r}_2+\boldsymbol{r}_1 \wedge \boldsymbol{t}_2)+\boldsymbol{r}_1 \wedge \boldsymbol{r}_2]\cdot \boldsymbol{t} = 0. \tag{4.2}$$

This quadratic in λ defines the two principal radii of curvature $\rho^{(1)}$, $\rho^{(2)}$ of the family of rays (see figure 3). The corresponding principal directions on $\mathscr{S}$ define a pair of orthogonal planes with respect to the ray direction (Weatherburn 1927, ch. 10). For the important special case in which the rays are normal to the reference surface $\mathscr{S}$, $\rho^{(1)}$ and $\rho^{(2)}$ are precisely the principal radii of curvature of $\mathscr{S}$.

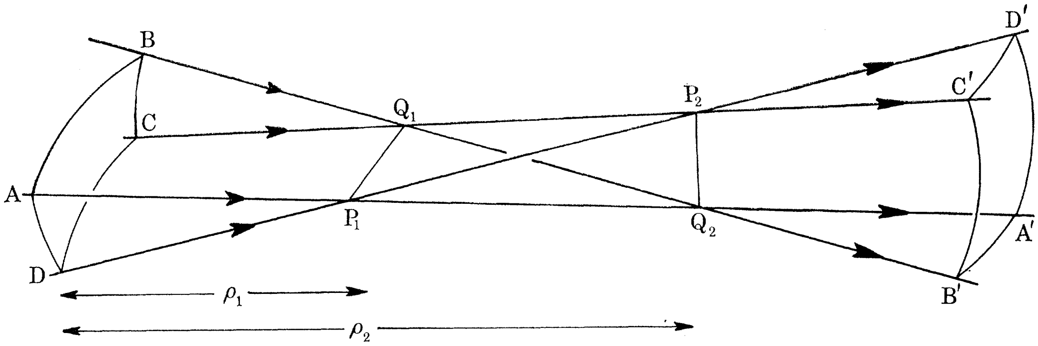

FIGURE 3. Intersection of adjacent ray paths in a uniform medium. The infinitesimal line segments P_1Q_1 and P_2Q_2 each lie in their respective sheet of the caustic surface and are perpendicular to each other.

The area of a ray tube can be expressed simply in terms of its initial area $\mathrm{d}A_0$, the distance from $\mathscr{S}$, and the principal curvatures

$$\mathrm{d}A(\xi^1, \xi^2, \lambda) = \mathrm{d}A_0(\xi^1, \xi^2)\ |1-\lambda/\rho^{(1)}|\ |1-\lambda/\rho^{(2)}|.$$

For steady infinitesimal waves the wave action equation (2.16) can be interpreted as a statement of the fact that the wave action flux $G_k a^2$ is conserved along ray tubes. Since in a uniform medium G_k is constant along rays, we can immediately find solutions for the wave amplitude in terms of the initial amplitude a_0 on the reference surface $\mathscr{S}$:

$$a(\xi^1, \xi^2, \lambda) = a_0(\xi^1, \xi^2)\ |1-\lambda/\rho^{(1)}|^{-\frac{1}{2}}\ |1-\lambda/\rho^{(2)}|^{-\frac{1}{2}}. \tag{4.3}$$

As has several times been noted, the linear ray solution becomes singular where adjacent rays meet. This caustic or focal surface $C^{(j)}$ has two sheets corresponding to the two principal radii $\rho^{(1)}$, $\rho^{(2)}$:

$$\boldsymbol{r}(\xi^1, \xi^2)+\rho^{(j)}(\xi^1, \xi^2)\ \boldsymbol{t}(\xi^1, \xi^2). \tag{4.4}$$

Sufficiently close to a caustic surface the distance along the ray direction can be expressed in terms of the radius of curvature α^{-1} of the caustic in the direction of the rays, and the distance X normal to the caustic (see figure 4). By this means it can be ascertained that the linear ray solution for the amplitude asymptotes to the singular expression:

$$a_0(\xi^1, \xi^2)\ \alpha^{\frac{1}{4}}\ |1/\rho^{(1)}-1/\rho^{(2)}|^{-\frac{1}{2}}\ |2X|^{-\frac{1}{4}} \quad \text{as} \quad |X| \rightarrow 0. \tag{4.5}$$

(*b*) *Near-linear theory*

In order to study the way in which weak nonlinearity modifies the 'slowly varying' solution near a caustic we make a local analysis. First a local Cartesian coordinate system with its origin on the caustic is introduced. The x-axis is taken to be along the normal and pointing away from the waves, and the y-axis to be in the direction of the grazing linear ray (see figure 4). Locally the caustic surface can be represented

$$x = \tfrac{1}{2}(\alpha y^2 + 2\beta yz + \gamma z^2) + \dots, \tag{4.6}$$

where α is positive, and cubic terms have been neglected.

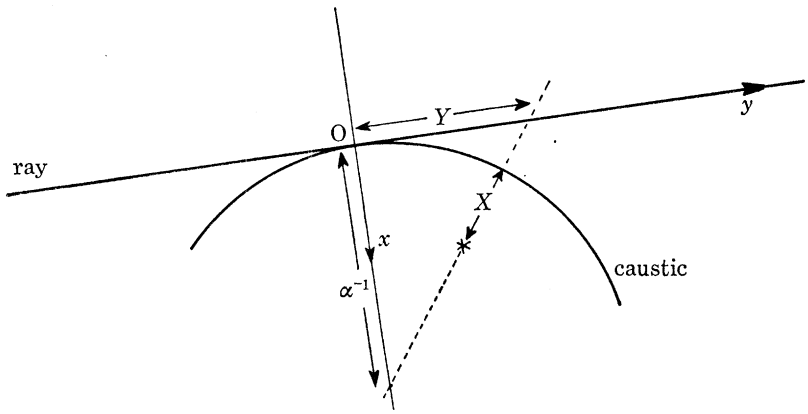

FIGURE 4. Definition of coordinates near a caustic surface. A ray Oy touches a caustic surface at O. Ox is the normal at O away from the waves, α is the curvature of the caustic surface in the plane Oxy.

Putting $\boldsymbol{k} = (l, m, n)$ and $\boldsymbol{k} = \boldsymbol{k}_0$ at the origin, the choice of Oy gives $G_l = G_n = 0$, and thus near the origin the leading terms in the dispersion relation (3.8) are

$$(m-m_0)\, G_m + \tfrac{1}{2}(l-l_0)^2\, G_{ll} + \tfrac{1}{2}(n-n_0)^2\, G_{nn} + a^2 H. \tag{4.7}$$

The derivatives of G are all independent of position since the medium is uniform. Again, H and its derivatives are assumed to be $O(1)$.

There are many ways of transforming coordinates so that the surface (4.6) coincides with a coordinate surface in the neighbourhood of the origin. For example, in two dimensions (i.e. $\beta = \gamma = 0$) coordinates corresponding either to a conformal transformation or to polar coordinates may be used. Here we do this by taking X to be the normal distance from the caustic and Y, Z to be the y, z coordinates of the intersection of the normal with the plane $x = 0$:

$$X = x - \tfrac{1}{2}(\alpha y^2 + 2\beta yz + \gamma z^2) + \dots, \tag{4.8}$$

$$Y = y + x(\alpha y + \beta z) + \dots, \quad Z = z + x(\beta y + \gamma z) + \dots. \tag{4.9}$$

It is convenient also to introduce transformed wavenumbers L, M, (i.e. the partial derivatives of the phase function χ with respect to X, Y, Z respectively):

$$\left.\begin{aligned} l &= L + \{(\alpha Y + \beta Z)\, M + (\beta Y + \gamma Z)\, N\} + \dots, \\ m &= M + \{-(\alpha Y + \beta Z)\, L + X(\alpha L + \beta N)\} + \dots, \\ n &= N + \{-(\beta Y + \gamma Z)\, L + X(\beta M + \gamma N)\} + \dots. \end{aligned}\right\} \tag{4.10}$$

In terms of the transformed coordinates and wavenumbers the leading terms in the consistency equation (2.11) and in the wave-action equation (2.16) can be written

$$\frac{\partial N}{\partial Y}-\frac{\partial M}{\partial Z}+\ldots = 0, \quad \frac{\partial L}{\partial Z}-\frac{\partial N}{\partial X}+\ldots = 0, \quad \frac{\partial M}{\partial X}-\frac{\partial L}{\partial Y}+\ldots = 0, \tag{4.11}$$

$$\frac{\partial}{\partial \boldsymbol{X}}\left(a^2 G_{\boldsymbol{k}}+\tfrac{1}{2}a^4 H_{\boldsymbol{k}}\right)+\ldots = 0. \tag{4.12}$$

For the linear case we already know that the 'slowly varying' solution for a and for $\partial l/\partial x$ becomes singular at the caustic. Thus, we anticipate that for the near-linear case there will be a similar rapid variation with X in both a and L as the caustic is approached. This leads us to make a boundary-layer approximation and to simplify equations (4.11, 12) by retaining only the X-derivatives. Also, there is now no loss of generality if we neglect the Y, Z-variation in the transformations (4.10) (the small change in the differential coefficients is precisely that of moving the origin of the local Cartesian coordinate system). It follows that M, N are locally constant and that the boundary layer form of the dispersion relation (4.7) is

$$X(\alpha m_0+\beta n_0)\, G_m+\tfrac{1}{2}(L-L_0)^2\, G_{ll}+a^2 H = 0. \tag{4.13}$$

The dominant terms in the wave-action equation are

$$X(\alpha m_0+\beta n_0)\, G_m+(L-L_0)\, G_{ll}+\tfrac{1}{2}a^2 H_l = -B/a^2. \tag{4.14}$$

As with equation (3.15) these terms may have differing orders of magnitude. On the wavy side of the caustic the boundary-layer solution matches with a linear ray solution of the far field provided that the constant of integration B is given by

$$B = a_0^2 \alpha^{\frac{1}{2}} |1/\rho^{(1)}-1/\rho^{(2)}|^{-1}\, [(\alpha m_0+\beta n_0)\, G_m G_{ll}]^{\frac{1}{2}}. \tag{4.15}$$

For curved caustics the role of the non-uniformity of the propagating medium, the term G_x, is taken over by the curvature term $(\alpha m_0+\beta n_0)\, G_m$. With this minor change, equations (4.13, 14) can be recognised as being similar to the approximations (3.14, 15) in the previous section. Thus, in the immediate vicinity of the origin the qualitative results are unchanged. In particular, there is the same two-fold classification of caustics according to the sign of H/G_{ll}, or of $(\alpha m_0+\beta n_0)\, G_m/H$.

Our choice of coordinate system implies that α is positive and that G_m is negative to conform with our convention that G_ω is positive. For two-dimensional waves, $\beta = 0$. Thus unless the waves are unusual and have m_0 negative, the type of caustic depends solely on the sign of H. For H positive, caustics will be of type S and for H negative of type R. Thus, all curved caustics in two-dimensions are of the same type for the same wave motion in a uniform medium.

Similar results hold for wave motion in an isotropic medium since in that case G may be written as

$$G(\omega, \boldsymbol{k}) = \omega^2-g(k) \tag{4.16}$$

and $\boldsymbol{k}$ is parallel, or anti-parallel, to the group velocity, so that l_0 and n_0 are zero.

5. Water waves on steady deep currents

A relatively simple example with a full range of different solutions is provided by deep-water waves on a current. A background to the subject is provided by the survey by Peregrine (1976) and further results are in P. & T.

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For deep-water waves the linear dispersion equation for still water is

$$G(\omega, \boldsymbol{k}) = \omega^2 - gk = 0, \tag{5.1}$$

where $k = |\boldsymbol{k}|$. (Note that use of Luke's (1967) Lagrangian leads to

$$\mathscr{L} = \tfrac{1}{4}\rho(\omega^2/gk - 1)\, a^2, \tag{5.2}$$

but any function $f(\omega, k)$ can be incorporated into a different measure of the amplitude. For the choice (5.1) of G the appropriate amplitude measure is proportional to the amplitude of the velocity potential for the motion.) For steady waves on a current $\boldsymbol{U}(\boldsymbol{x})$ the Doppler relation,

$$\omega = \sigma + \boldsymbol{k}\cdot\boldsymbol{U}, \tag{5.3}$$

where σ is the frequency of waves relative to the water, gives

$$G(\omega, \boldsymbol{k}, \boldsymbol{x}) = (\omega - \boldsymbol{k}\cdot\boldsymbol{U})^2 - gk = 0 \tag{5.4}$$

as the dispersion equation for linear waves on a deep current.

If we put
$$\boldsymbol{U}(\boldsymbol{x}) = U(\boldsymbol{x})\,\boldsymbol{i} + V(\boldsymbol{x})\,\boldsymbol{j}, \tag{5.5}$$
and
$$\boldsymbol{k} = l\boldsymbol{i} + m\boldsymbol{j}, \tag{5.6}$$

then at a straight caustic, $x = 0$, caused by the current we have equation (5.4) and

$$G_l = -2U(\omega - lU - mV) - gl/k = 0. \tag{5.7}$$

The simplifying assumption of a straight caustic means that it is only the x dependence of $\boldsymbol{U}$, which is taken into account close to the caustic. The derivation in § 3 corresponds to assuming $\boldsymbol{U}(x)$, ω and m are known and using (5.4) and (5.7) to determine the caustic position and the value, l_0, of l at the caustic. However, it is more interesting to examine these equations from another direction.

That is, we enquire, 'on what currents can caustics occur?' This is answered by choosing m, the wave number component along the caustic and eliminating l between equations (5.4) and (5.7), which yields a relation between $U\omega/g$ and $V\omega/g$. Then, for each mg/ω^2 there is a line of points in the $\boldsymbol{U}\omega/g$ plane at which caustics can occur. There is no loss of generality in choosing m and ω to be > 0 and figure 5 shows some lines of caustic points in the $\boldsymbol{U}\omega/g$ plane.

Each line of caustics points, other than $m = 0$, has two symmetrical branches. The point of symmetry is

$$\boldsymbol{U} = (0, \omega/m). \tag{5.8}$$

The cusps correspond to the triple-root caustics (i.e. $G = G_l = G_{ll} = 0$) discussed in the appendices. On each branch l varies smoothly from $-\infty$, through zero at $U = 0$, to $+\infty$. The line $U = 0$ also satisfies equations (5.4) and (5.7) as $k \to \infty$. This is a stronger singularity than a caustic and is discussed further in P. & T. (It corresponds to $l \to \infty$ in figure 7.)

The way in which these caustics arise is clarified by considering two specific simple current distributions:

$$(a)\quad \boldsymbol{U}(\boldsymbol{x}) = V(x)\,\boldsymbol{j}, \tag{5.9}$$

and

$$(b)\quad \boldsymbol{U}(\boldsymbol{x}) = U(x)\,\boldsymbol{i}. \tag{5.10}$$

Case (*a*) corresponds to a shearing current which bends rays around until they are parallel to the current and form a caustic if the current increases sufficiently with x. Figure 6 shows solutions of the dispersion equation for various values of m. Note how two different caustics

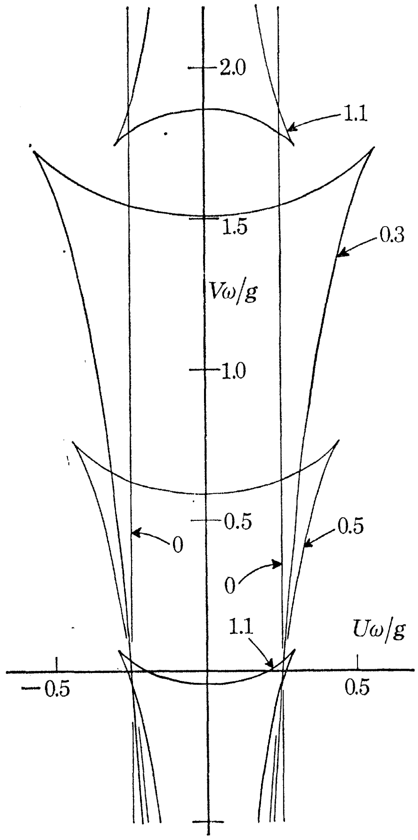

FIGURE 5. Currents on which a steady straight caustic may form for periodic wave trains. Each line is for a different value of mg/ω^2.

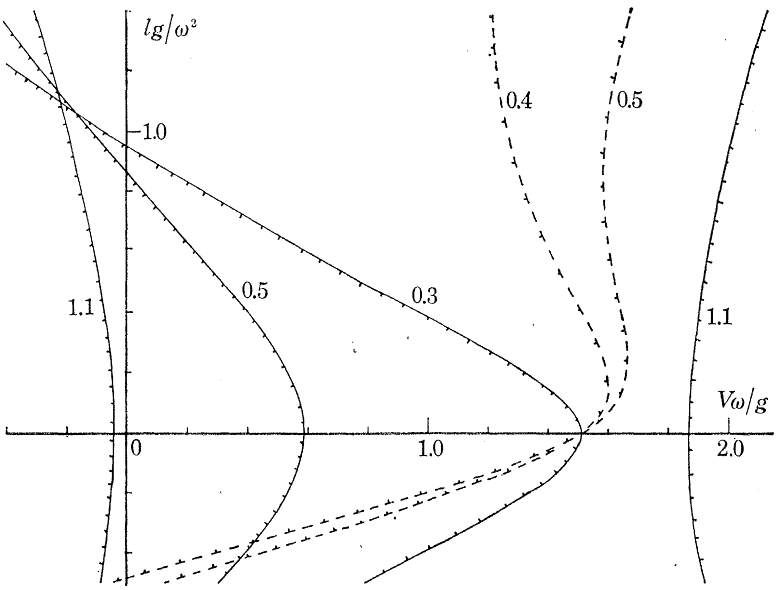

FIGURE 6. Linear dispersion relations for waves on a current $U\boldsymbol{i}+V(x)\boldsymbol{j}$. The full lines are for $U = 0$ and are labelled with values of mg/ω^2. The broken lines are for $mg/\omega^2 = 0.3$ and are labelled with values of U. The marks on each line lie on the same side as near-linear solutions. Caustics occur where the lines are vertical.

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can arise for one value of m if $V(x)$ has sufficient variation. (A caustic occurs where the lines have vertical tangents.) These would involve quite different wave trains. The incident wave train is simply reflected with a change of sign of l. The same figure also includes solutions for non-zero U to show how a further different caustic arises.

Case (*b*) corresponds to currents flowing with or against the waves. In the simplest case, $m = 0$, a caustic occurs when an adverse current (i.e. $U < 0$) is sufficiently strong to stop the waves propagating. The 'stopping' velocity is $U = -\frac{1}{4}g/\omega$. In this case if

$$1 < mg/\omega^2 < \tfrac{3}{4}\sqrt{2} \tag{5.11}$$

two different caustics can arise for any wave train, as may be seen from figure 7. Peregrine (1976, § IID) gives more details of this case as well as some details of the linear solutions for the caustics for $m = 0$.

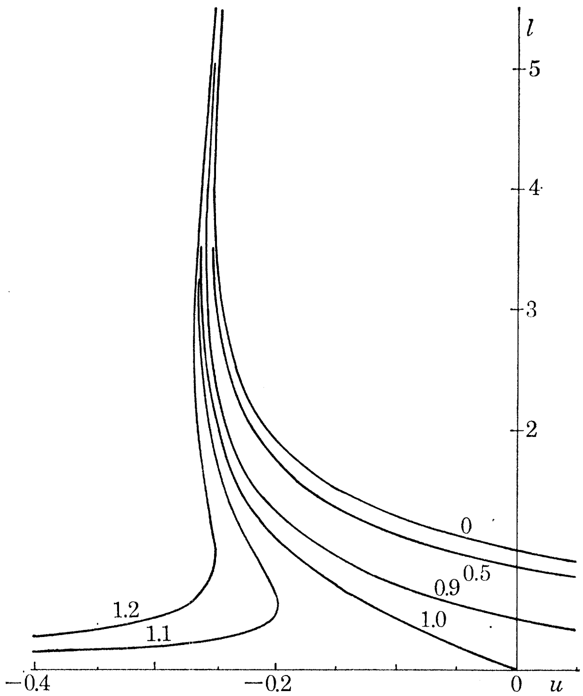

FIGURE 7. Part of the linear dispersion relation for waves with wave-number vector $\boldsymbol{k} = l(x)\,\boldsymbol{i}+m\boldsymbol{j}$ on a current $U(x)\,\boldsymbol{i}$. The values of mg/ω^2 are given on each line; $u = U\omega/g$; the near-linear solution lies to the left of each line. Comparison may be made with figure 4 of P. & T. which shows finite-amplitude solutions for $m = 0$.

The near-linear dispersion equation for water waves on deep currents is

$$G+Ha^2 = (\omega-\boldsymbol{k}\cdot\boldsymbol{U})^2-gk(1+k^2a^2) = 0. \tag{5.12}$$

In figures 6 and 7 an indication is given of which side of the linear dispersion relation it lies. The type of near-linear caustic depends on whether the linear curve is concave or convex toward its 'near-linear side' at the caustic point. It may be seen that both types of caustic occur. The caustic on a simple shear is of type R, but the stopping velocity caustic is of type S.

When these are related to the caustic lines in figure 5 we find that type R caustics occur between the cusps and type S caustics occur on the branches of the lines which extend to

$V = \pm\infty$. No detailed near-linear solutions have been produced since P. & T. give details of finite-amplitude solutions for a representative caustic of each type. (These solutions were available before this theory was elucidated.)

6. Uniform approximation for linear waves at a caustic

Section 3 illustrates the singular behaviour of a simple slowly-varying-wave approximation at a caustic. The linear wave theory itself is not singular, either at a caustic or a focus. It is well over a century since wave behaviour near a caustic was first effectively analysed (Airy 1838). Discussion usually starts from an integral formulation of the problem (see, for example, Lighthill 1978) which is particularly appropriate for those cases where the inhomogeneities in the waves are caused by the initial conditions or are produced in a small finite region. However, often the wave propagation is entirely in an inhomogeneous region, for example waves of fluid motion propagating in the ocean or atmosphere. In this case a differential approach which can be added, as a local approximation, after calculating rays is likely to be more effective.

The extra terms which are needed to describe caustics and which do not appear in the basic theory outlined in the previous sections are called 'diffraction' terms by Hayes (1973) and 'higher-order dispersive' terms by Whitham (1974). Both descriptions are suitable, but since 'diffraction' is used in other senses, the latter description is used here and is abbreviated to 'h.o.d.'

There are several ways in which the h.o.d. terms may be derived. The most satisfactory derivatives include careful expansions in terms of a small parameter, and corresponding multiple time and space scales (Smith, 1976*a*). Here, we aim for a quick simple derivation. A heuristic Taylor expansion of an operator is used. Similar devices have been used for many years (for example by Rayleigh in 1876, see Lamb 1932, § 252; Benney & Newell 1967; Davey 1972). It is clear that this method may be justified rigorously for simple linear partial differential operators on suitably well-behaved functions, but it would be interesting to know the limits to its use, for example, (*a*) for complicated linear operators, such as that for water waves in water of depth h, which gives

$$G(\omega, k) = \omega^2 - gk\tanh kh, \tag{6.1}$$

and (*b*) for nonlinear operators.

The linear equations for wave propagation can be written in a form

$$\mathscr{G}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \boldsymbol{x}}, \boldsymbol{x}\right)\phi = 0, \tag{6.2}$$

where $\mathscr{G}$ is a linear operator. For a single-wave solution we can identify this equation with its 'Fourier transform'

$$G(\omega, \boldsymbol{k}, \boldsymbol{x}, t)\, a\mathrm{e}^{\mathrm{i}\chi} = 0, \tag{6.3}$$

where $G = 0$ is the dispersion equation and its $(\boldsymbol{x}, t)$ dependence is on a longer scale than that of χ. Rapid variations are taken to be entirely in

$$\chi = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t.$$

Now, for a local solution in the neighbourhood of a point $\boldsymbol{x}_0$ where $\omega = \omega_0$ and $\boldsymbol{k} = \boldsymbol{k}_0$, and the amplitude is considered to vary slowly we identify

$$\boldsymbol{k} \quad \text{with} \quad \boldsymbol{k}_0 - \mathrm{i}\partial/\partial\boldsymbol{x},$$

and

$$\omega \quad \text{with} \quad \omega_0 + \mathrm{i}\partial/\partial t,$$

so that equation (6.3) becomes

$$G\left(\omega_0 + \mathrm{i}\frac{\partial}{\partial t}, \mathbf{k}_0 - \mathrm{i}\frac{\partial}{\partial \boldsymbol{x}}, \boldsymbol{x}\right) a(\boldsymbol{x}, t)\, \mathrm{e}^{\mathrm{i}\chi_0} = 0. \tag{6.4}$$

By allowing slow changes in phase to be accounted for by complex values of a and by assuming that a Taylor expansion of the function/operator G is valid about $(\omega_0, \boldsymbol{k}_0, \boldsymbol{x}_0)$, equations (6.4) becomes

$$Ga + \mathrm{i}G_\omega \frac{\partial a}{\partial t} - \mathrm{i}G_k \cdot \frac{\partial a}{\partial \boldsymbol{x}} + (\boldsymbol{x} - \boldsymbol{x}_0)\cdot G_{\boldsymbol{x}} a - \tfrac{1}{2}G_{\omega\omega}\frac{\partial^2 a}{\partial t^2} - G_{\omega k}\cdot \frac{\partial^2 a}{\partial \boldsymbol{x}\,\partial t} - \tfrac{1}{2}G_{kk} : \frac{\partial^2 a}{\partial \boldsymbol{x}\,\partial \boldsymbol{x}} + \dots = 0, \tag{6.5}$$

in which G and its derivatives are evaluated at $(\omega_0, \boldsymbol{k}_0, \boldsymbol{x}_0)$.

For the region close to a straight caustic, $x = 0$, of the type considered in § 3, G_l is zero and the amplitude only varies with x. Hence equation (6.5) reduces to

$$xG_x a - \tfrac{1}{2}G_{ll}\,\mathrm{d}^2 a/\mathrm{d}x^2 = 0, \tag{6.6}$$

for which the appropriate solution is

$$a = A_0 \,\mathrm{Ai}\,(\rho), \quad \text{with} \quad \rho = x(2G_x/G_{ll})^{\frac{1}{3}}. \tag{6.7}$$

Similar Airy function solutions result when the caustic is curved. The complex constant A_0 is found by matching the Airy function with the ray solution by using the asymptotic formula:

$$\mathrm{Ai}\,(\rho) \simeq -\pi^{-\frac{1}{2}}(-\rho)^{-\frac{1}{4}} \sin\left[\tfrac{2}{3}(-\rho)^{\frac{3}{2}} + \tfrac{1}{4}\pi\right], \tag{6.8}$$

for large negative values of ρ.

The range of applicability of an Airy function representation can be greatly extended if the amplitude A and argument ρ are regarded as functions of position to which the above results are the one-term Taylor-series approximation (Ludwig 1966). The asymptotic formula (6.8) permits the uniformly valid Airy function solution to be determined from the singular ray solutions for the waves incident on and reflected from the caustic. Thus the singular representation

$$u = a^{(1)} \exp\,(\mathrm{i}\chi^{(1)}) + a^{(2)} \exp\,(\mathrm{i}\chi^{(2)}), \tag{6.9}$$

where $^{(1)}$ and $^{(2)}$ denote incident and reflected wave trains, is replaced by (or smoothly matches with) the uniform representation

$$u = \{A\,\mathrm{Ai}\,(\rho) + \mathrm{i}C\,\mathrm{Ai}'\,(\rho)\} \exp\,(\mathrm{i}\sigma), \tag{6.10}$$

where

$$\left.\begin{aligned} \rho &= -\left[\tfrac{3}{4}(\chi^{(1)} - \chi^{(2)})\right]^{\frac{2}{3}}, \quad \sigma = \tfrac{1}{2}(\chi^{(1)} + \chi^{(2)}), \\ A &= \pi^{\frac{1}{2}}(-\rho)^{\frac{1}{4}}\,(a^{(1)} + a^{(2)}), \quad C = \pi^{\frac{1}{2}}(-\rho)^{-\frac{1}{4}}\,(a^{(1)} - a^{(2)}). \end{aligned}\right\} \tag{6.11}$$

For a number of types of waves the computer calculation of ray paths, and hence of the linear ray solutions, has become routine. Careful programming permits the ray solution to be continued after any reflection at caustics. To adapt such a computer calculation to remain valid at caustics merely requires a sub-routine to evaluate Airy functions, and the use of equations (6.10 and 6.11). Some numerical smoothing may be desirable since in the immediate vicinity of the caustic the numerical values of the ray quantities may be unreliable, but it is known that the derived quantities ρ, σ, A, C vary smoothly.

7. Higher-order-dispersive effects for near-linear waves

Near-linear terms can be included in the analysis of the previous section by noting that usually the dispersion equation has the form

$$G+H|a|^2 = 0. \tag{7.1}$$

The partial derivative of this expression with respect to $|a|^2$ is H and so the first term in its Taylor expansion about $|a|^2 = 0$ is $H|a|^2$ which gives a plausible reason for adding a term $H|a|^2$ on to equation (6.5). The plausibility of this latter move is increased by considering equation (6.5) with this nonlinear term for a uniform medium, i.e. $G_x = 0$, and in a frame of reference moving with the linear group velocity. Then $G = 0$ and $G_k = 0$. The equation that results from keeping only the most important time and spatial derivatives is

$$\mathrm{i}G_\omega \frac{\partial a}{\partial t} - \tfrac{1}{2}G_{\boldsymbol{kk}} : \frac{\partial^2 a}{\partial \boldsymbol{x}\,\partial \boldsymbol{x}} + H|a|^2 a = 0. \tag{7.2}$$

This equation is the now well-known nonlinear Shrödinger equation for modulated dispersive waves (Whitham 1974, ch. 17). The second derivative terms in (7.2) represent the effect of h.o.d.

By adding the near-linear terms to the equation (6.6) for waves near a straight caustic, we obtain

$$-\tfrac{1}{2}G_{ll}\frac{\mathrm{d}^2 a}{\mathrm{d}x^2} + xG_x a + H|a|^2 a = 0. \tag{7.3}$$

This can be transformed into the standard form chosen by Smith (1976*a*) by introducing ρ and A_0 as used in the linear h.o.d. solution (6.7) to give

$$\mathrm{d}^2\,\mathrm{Ai}\,(\rho;\beta)/\mathrm{d}\rho^2 - \rho\,\mathrm{Ai}\,(\rho;\beta) + \beta\,\mathrm{Ai}^3\,(\rho;\beta) = 0, \tag{7.4}$$

where
$$a = A_0\,\mathrm{Ai}\,(\rho;\beta) \quad \text{and} \quad \beta = -\frac{2H}{G_{ll}}A_0^2\left(\frac{G_{ll}}{2G_x}\right)^{\frac{2}{3}}. \tag{7.5}$$

Miles (1978*a*, *b*) also meets this equation and chooses the standard form

$$\frac{\mathrm{d}^2 A}{\mathrm{d}\rho^2} - \rho A \pm 2A^3 = 0, \tag{7.6}$$

where
$$A = |\tfrac{1}{2}\beta|^{\frac{1}{2}}\,\mathrm{Ai}\,(\rho;\beta) = a|H/2G_x|^{\frac{1}{3}}|H/G_{ll}|^{\frac{1}{6}}. \tag{7.7}$$

The most suitable choice of normalisation depends upon the physical context. The first standard form (7.4), together with the far field condition (7.8) given below, is appropriate for wave reflexion. Particular emphasis is given to the oscillatory side of the solution and to the close connection with Airy functions (hence the suggestive notation). On the other hand, the second standard form (7.6) with the boundary condition

$$A \simeq \epsilon\,\mathrm{Ai}\,(\rho) \quad \text{as} \quad \rho \to \infty,$$

emphasises the non-oscillatory side of the solution. For $\epsilon < 1$ the two families of solutions are the same with

$$\beta = 2\ln(1 \pm \epsilon^2).$$

However, for the '$-$' case with $\epsilon > 1$ the solutions of equation (7.6) are non-oscillatory and become unbounded as ρ tends to minus infinity.

The solutions of these equations are Painlevé transcendents. They are real functions which is why the modulus sign was dropped in the standard forms (7.4 and 7.6). The magnitude of the nonlinear effects near the caustic is indicated by the value of β. It is only for extremely large β that there is any substantial variation in the maximum wave amplification from that of the linear Airy function solution. The most marked effect is that there is phase shift of the wave profile. This reveals itself both in the far-field solution

$$\mathrm{Ai}\,(\rho;\beta) \simeq \pi^{-\frac{1}{2}}(-\rho)^{-\frac{1}{4}}\sin\left[\tfrac{2}{3}(-\rho)^{\frac{3}{2}}+(\tfrac{3}{8}\beta/\pi)\ln|\rho|+\text{const.}\right], \tag{7.8}$$

and in the form of the differential equation

$$\mathrm{d}^2\,\mathrm{Ai}\,(\rho;\beta)/\mathrm{d}\rho^2-[\rho-\beta\,\mathrm{Ai}^2\,(\rho;\beta)]\,\mathrm{Ai}\,(\rho;\beta) = 0. \tag{7.9}$$

By comparing this equation with the Airy equation an indication of the shift is obtained by setting the term in square brackets to zero. The direction of the shift is seen to depend on the sign of β which, since it depends on the sign of H/G_{ll}, depends on whether the caustic is of type R or S. For type R caustics, $\beta < 0$, and the waves do not reach as far as in the linear solution. For type S caustics, $\beta > 0$, and waves penetrate further.

Some further analysis is possible for the unsteady near-linear case, for which a canonical form of equation (7.2) is

$$\mathrm{i}\,\frac{\partial A}{\partial T}+\frac{\partial^2 A}{\partial X^2}-XA\pm 2|A|^2A = 0, \tag{7.10}$$

where

$$T = t\,\frac{G_x}{G_\omega}\left(\frac{G_{ll}}{2G_x}\right)^{\frac{1}{3}}. \tag{7.11}$$

Kaup & Newell (1978, § 6) show that a further transformation of variables

$$\xi = X+T^2, \quad \tau = T, \tag{7.12}$$

$$\phi(\xi,\tau) = A(X,T)\exp\left[\mathrm{i}(XT+\tfrac{1}{3}T^3)\right], \tag{7.13}$$

gives the nonlinear Schrödinger equation

$$\mathrm{i}\,\frac{\partial\phi}{\partial\tau}+\frac{\partial^2\phi}{\partial\xi^2}\pm 2|\phi|^2\phi = 0. \tag{7.14}$$

The nonlinear Schrödinger equation has been studied intensively in the last few years (see references in Kaup & Newell 1978). Solutions of particular importance are the 'envelope soliton'

$$\phi = \mu\,\mathrm{sech}\,(\mu\xi 2^{-\frac{1}{2}})\exp\,(\mathrm{i}\mu^2\tau) \tag{7.15}$$

for the '+' equation, and the 'envelope hole'

$$\phi = \mu\tanh\,(\mu\xi 2^{-\frac{1}{2}})\exp\,(-2\mathrm{i}\mu^2\tau) \tag{7.16}$$

for the '−' equation. The corresponding solutions $A(X,T)$ do not appear to be of particular value in these cases. They show none of the variation in amplitude one would expect to find associated with the caustic since

$$|A| = |\phi|, \tag{7.17}$$

and the amplitudes are independent of time.

The expression $(X+T^2) = \xi$ may be thought of as the primary phase function of the modulation since ξ = constant describes the path of a soliton into and away from the caustic.

However, the apparent frequency and wavenumber of the modulation in these solutions are

$$-\partial\psi/\partial T = -\mu^2+(X+T^2) \quad \text{or} \quad 2\mu^2+(X+T^2), \tag{7.18}$$

and

$$\partial\psi/\partial x = -T, \quad \text{where} \quad \psi = \arg(A). \tag{7.19}$$

Thus, for both the '+' and '−' cases, there is secular variation with ξ and T of the phase modulation. It may be that just this variation is necessary to maintain the constant amplitude of the soliton. Similar criticisms apply to the more complicated explicit solutions of the non-linear Schrödinger equation which describe multiple soliton interactions. For more discussion, see Newell (1978).

8. Conclusion

Linear and near-linear waves in the vicinity of a caustic have been examined. Solutions for linear waves, including expressions for matching them, have been derived in a succint manner so that applications are straightforward. For near-linear waves the existence of two different types of behaviour near caustics have been exposed. The h.o.d. terms give equations which permit uniform solutions at caustics, but although one finds non-singular solutions it does not follow that the true solution is found to a particular physical problem.

The companion paper, P. & T., shows that when finite-amplitude solutions are used, rather than small-amplitude near-linear solutions, a different picture emerges for a type S caustic. The example studied in P. & T. is for waves meeting a directly adverse current. Solutions are found which at all points satisfy a requirement that they be slowly varying, so there is no need to look for h.o.d. effects. But these solutions show the wave amplitude growing right up to the maximum for which plane wave solutions exist. In this particular physical context we then expect waves to break. Conversely for a type R caustic slowly varying solutions become singular at quite small amplitudes so that it is to be expected that h.o.d. solutions apply. Thus we come to the tentative conclusion that (i) type R caustics are regular, smooth solutions can be found and the wave field has no singularity, and (ii) type S caustics may be singular and if waves can break in some sense, as is the case for surface waves and internal gravity waves, then they may do so.

There appears to be a similarity between type S caustics and water waves incident on a beach. A simple linear 'ray' theory would give a singularity at the shoreline as the wavelength approached zero with the depth. However, more detailed linear analysis gives uniform solutions which correspond to perfectly reflected waves. For very low steepness waves this is what is observed, except for viscous effects. If a shallow-water (corresponding to slowly varying) nonlinear theory is used then waves above a certain steepness exhibit a singularity of the approximation. Meyer & Taylor (1972) give explicitly the bounding amplitude of perfect reflexion in this theory. If a uniform near-linear approximation is used (e.g. the Boussinesq equations as in Peregrine 1967) then we do not expect solutions to have singularities, though at present such a theory has not been extended to the shoreline. However, we know that in practice all but the gentlest waves break on a beach and there is usually little reflexion (less than 10% is normal).

If the tentative conclusion that R caustics are regular and S caustics can be composed of strong nonlinear waves is accepted, the analysis of curved caustics in § 4 is particularly valuable in answering qualitative questions such as 'are strongly non-linear waves likely to arise?' in

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particular circumstances. Strongly nonlinear waves are likely to arise at S type caustics and for wide classes of wave propagation the type of caustic depends only on the sign of the function H in the dispersion relation (2.15).

For deep-water gravity surface waves H is negative so that curved caustics are of type R. For capillary-gravity waves (Wilton 1915, or see Lighthill 1967, eqn (89)), H changes sign at the wavenumber corresponding to the minimum group velocity $k^2 = \frac{1}{2}\rho g/\gamma$, where γ is the surface tension. Thus strong nonlinear effects can be expected near curved caustics of capillary waves, if viscous effects do not dominate.

For nonlinear optics, Whitham (1974, eqn 16.11) gives the dispersion relation

$$\omega^2 - \frac{\omega^2 \nu_p^2}{\omega^2 - \nu_0^2} - c_0^2 k^2 + \frac{3\alpha\epsilon_0^2 \nu_p^6 \omega^2}{4(\omega^2 - \nu_0^2)^4} a^2 + \ldots = 0 \tag{8.1}$$

for near-linear propagation of electromagnetic waves in a dielectric. For the normal case in optics, where $\alpha > 0$ and $\omega^2 < \nu_0(\nu_0 - \nu_p)$, H is positive and curved caustics may be strongly nonlinear. The physical event corresponding to breaking of water waves is probably the electrical or mechanical breakdown of the propagating medium.

In all these cases, there is little point considering behaviour at a caustic, if waves become too strongly nonlinear *before* the caustic is reached. This clearly occurs in some solutions given by P. & T. for water waves on an adverse current. This is also possible for waves approaching a curved, R-type, caustic. A paper on finite-amplitude water waves approaching such a caustic is in preparation.

Four major points have not been treated here. One is the matching of an h.o.d. near-linear solution at a caustic with a near-linear slowly-varying solution away from a caustic. This problem is twofold. Both the near-linear equations and the solution not too far away from the caustic must involve both the incident and reflected wave trains. That is it may be based on an averaged Lagrangian of the form

$$\mathscr{L} = G_1 a_1^2 + G_2 a_2^2 + \tfrac{1}{2}H_1 a_1^4 + \tfrac{1}{2}H_2 a_2^4 + I_{12} a_1^2 a_2^2, \tag{8.2}$$

where suffices 1 and 2 refer to the two wave trains and $I_{12}(\boldsymbol{k}_1, \boldsymbol{k}_2)$ comes from the interaction of the wave trains. For example for deep water waves one might use the results from Longuet-Higgins & Phillips (1962), from Willebrand (1975) or from Weber & Barrick (1977).

The second, more difficult, problem is how to determine when an h.o.d. near-linear solution to a type S caustic is invalid. It is possible that one simply needs to check that the predicted maximum wave amplitude is within reasonable bounds, but the results of P. & T. do not give much confidence that this will be adequate.

The third point, for which further analysis is required, is a treatment of waves, such as near-linear water waves in finite-depth of water, which need further variables arising from non-periodicity of a potential function.

The fourth topic is the behaviour of non-dispersive waves for which some results are given by Ostrovsky (1976) and Bobbit & Cumberbatch (1976).

The bulk of this paper has concerned caustics, that is the simplest type of singularity of the linear ray solution. Higher-order singularities can be expected to be met less often, but this reduced frequency of occurrence is partially countered by greater wave intensification. The locally large amplitudes mean that studies of nonlinear effects are particularly pertinent.

Higher-order singularities may be viewed as coalescences of caustics, and the necessary mathematical methods are an extension of those developed for caustics. There are three illustrations of this extension in the appendices.

The first, appendix A, considers near-linear effects near a straight 'triple-root' singularity; that is where two caustics coalesce with the dispersion relation having a triple root in wave number. This may be contrasted with the simpler case where the *spatial* variation corresponds to two coalescing caustics; see McKee (1975), Peregrine & Smith (1975) and Stiassnie & Dagan (1978) for water-wave examples. Near-linear and h.o.d. effects for the triple-root caustic are considered in appendix C.

Appendix B gives a uniform linear approximation for a cusped caustic. During the preparation of this paper we learned that this repeats work of Holford (1974) but feel that since only a very brief abstract was published, more detail would be of value.

A brief account of some of this work was presented at the I.U.T.A.M. symposium on 'Waves on water of variable depth' at Canberra, July 1976 (Peregrine & Thomas 1976) and other parts were presented at a conference of the Institute of Mathematics and its Applications in June 1977 by R. Smith. During this collaboration R.S. was supported by a fellowship from C.E.G.B.

Appendix A. Nonlinear effects near a 'triple root' caustic

For one-dimensional linear wave propagation a triple root is defined to be the circumstance in which there is a three-fold coalescence of roots of the dispersion relation. If $G(x, \boldsymbol{k})$ has a Taylor series expansion, then at the triple root

$$G(0, \boldsymbol{k}_0) = G_l(0, \boldsymbol{k}_0) = G_{ll}(0, \boldsymbol{k}_0) = 0, \tag{A 1}$$

where the triple-root position is taken to be $x = 0$. Thus, locally we have

$$G(x, \boldsymbol{k}) = xG_x(0, \boldsymbol{k}_0) + \tfrac{1}{6}(l-l_0)^3\, G_{lll}(0, \boldsymbol{k}_0) + \dots . \tag{A 2}$$

In order to achieve compatibility with the earlier conventions that the waves are incident from the $-x$ direction, and that G_ω is positive, it is necessary that G_{lll} be negative.

As noted in § 3, the fact that for linear waves the dispersion relation is uncoupled from the wave action equation makes it particularly easy to investigate the nature of any singularities. The dispersion relation (A 2) can be solved to give

$$l \approx l_0 - (6G_x/G_{lll})^{\frac{1}{3}}\, x^{\frac{1}{3}}, \tag{A 3}$$

and from the wave action equation (3.2) it follows that

$$a \approx (-2B^3/9G_x^2 G_{lll})^{\frac{1}{6}}\, x^{-\frac{1}{6}}.$$

For near-linear waves the results (3.8–3.12) are valid independently of the presence or character of any singularities. Thus in the (x, l) plane the near-linear solution curves are to the left or right of the linear-theory dispersion relation, $G = 0$, according to whether H/G_x is positive or negative. Also, for small wave-action flux, B, the solution curves remain close to, but cannot cross, the two dividing lines

$$G = 0 \quad \text{and} \quad HG_l - \tfrac{1}{2}GH_l = 0. \tag{A 5}$$

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The particular character of the singularities is revealed when we come to sketch the curves (A 5).

For triple roots these curves have the local approximations

$$x(G_{lll}/G_x) = -\tfrac{1}{6}(l-l_0)^3 \quad \text{and} \quad x(G_{lll}/G_x) = (H/H_l)\,(l-l_0)^2. \qquad \text{(A 6, A 7)}$$

At the triple-root position there is a two-point contact but the curves do not cross. If H_l is negative, solution lines are on the 'other side' of line (A 6)–(A 7) and so can smoothly pass through the singular region, see figure 8(*a*), (*b*). However, despite their smoothness singular

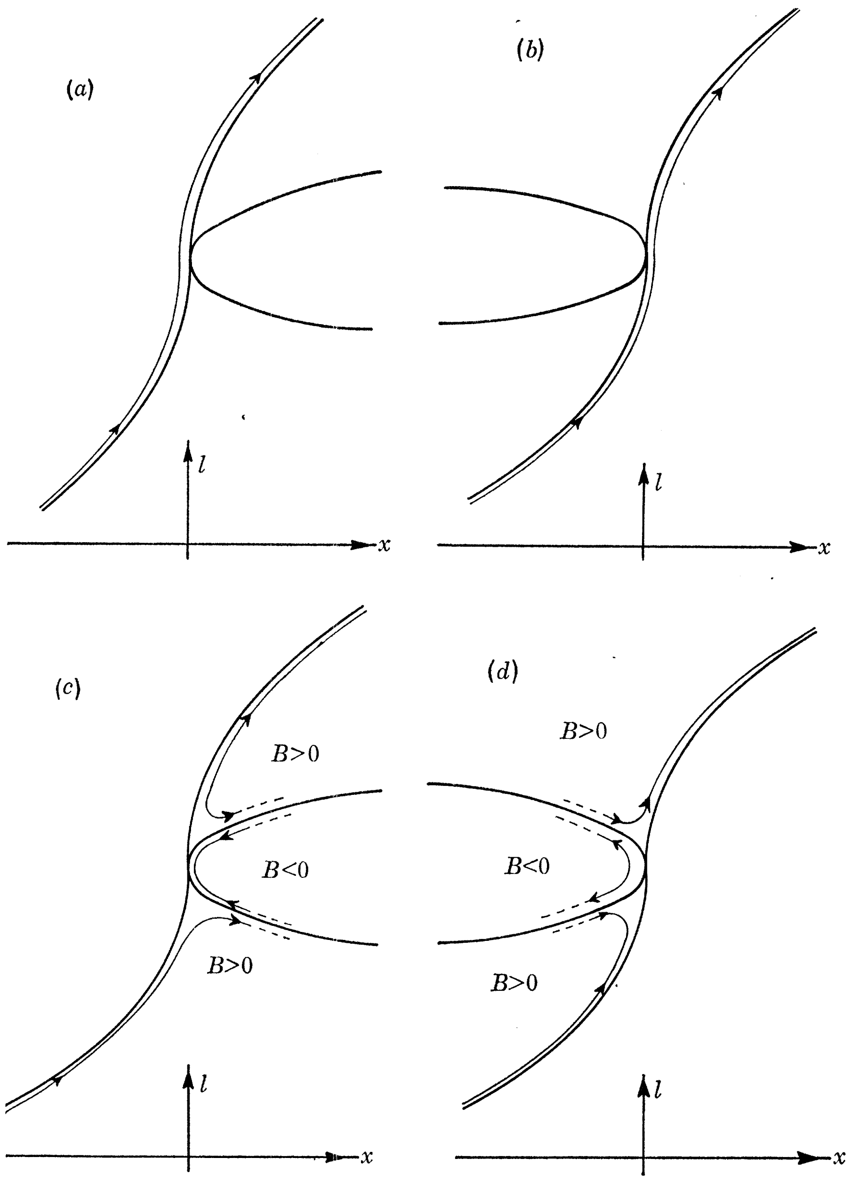

FIGURE 8. Diagrams illustrating the behaviour of near-linear waves close to a 'triple-root' point. Heavy lines represent the solutions (A 5) for $B = 0$. The thin lines sketch out possible near-linear solutions. The direction of wave propagation ($+x$ or $-x$) on each solution branch is indicated by arrows, and the lines are broken where they depart far from the linear solution. All the diagrams are for $G_x > 0$. For $G_x < 0$ reflect the diagrams about a horizontal line through the triple-root point. (*a*) $H > 0$, $H_l < 0$. (*b*) $H < 0$, $H_l < 0$. (*c*) $H < 0$, $H_l > 0$. (*d*) $H > 0$, $H_l > 0$.

gradients are possible so that the obvious deduction of transmission of the wave through the singularity (T – type?) needs to be verified by an h.o.d. calculation.

On the other hand if H_l is positive there are five solution branches, see figure 8 (*c*), (*d*) and the incident wave may be on a branch similar to that for an S-type or an R-type caustic. Indeed the triple root caustic should be envisaged as the coalescence of an R-type and an S-type caustic, e.g. at the cusp points in figure 5.

Caustics may coalesce because of a spatial variation affecting the dispersion relation. That is, if $G_x = 0$ at a caustic. In these circumstances waves may be trapped, or partially reflected; see McKee (1975), Peregrine & Smith (1975) and Stiassnie & Dagan (1978) for linear examples.

Appendix B. Uniform linear approximation for cusped caustics

Although the linear ray solution is not valid at caustics or higher-order ray coalescences, a uniformly valid solution can be constructed by means of a canonical superposition of ray solutions (Ludwig 1966)

$$u = \int_\Gamma [\mathscr{A}(\boldsymbol{x}, \boldsymbol{y}, \xi) - \mathrm{i}\mathscr{B}(\boldsymbol{x}, \boldsymbol{y}, \xi)] \exp(\mathrm{i}\chi(\boldsymbol{x}, \xi))\, \mathrm{d}\xi. \tag{B 1}$$

Here $\mathscr{A}$ is an in-phase amplitude term, χ the rapidly varying phase, $\mathscr{B}$ is an out-of-phase amplitude associated with the transverse (or modal) $\boldsymbol{y}$-structure of the wave and ξ is a ray parameter, as in § 4. Away from the region of coalescence the integral can be approximated by the method of stationary phase:

$$u \simeq \sum_j (2\pi)^{\frac{1}{2}} |\partial_\xi^2 \chi^{(j)}|^{-\frac{1}{2}} |\mathscr{A}^{(j)} - \mathrm{i}\mathscr{B}^{(j)}| \exp(\mathrm{i}\chi^{(j)}), \tag{B 2}$$

where

$$\partial_\xi \chi = 0 \quad \text{at} \quad \xi = \xi^{(j)}. \tag{B 3}$$

This is precisely the linear ray solution, with due allowance for distinct ray paths going through the one physical point. Thus, as has often been noted, once the linear h.o.d. solution is known it is straightforward to determine the non-uniform ray solution. The purpose of this appendix is to show that for cusped caustics the reverse is also true. Namely, that the singular ray solutions provide sufficient information to permit the construction of the uniform solution.

The various types of ray coalescences can be catalogued in terms of the change in the number of real roots of the stationary phase condition (B 3). Thus, as canonical forms for χ it is natural to choose simple polynomials in ξ with the required root behaviour. Equivalently, we can make direct use of the standard forms provided by Catastrophe Theory (Thom 1975, Berry 1976).

For a caustic there are two roots before the reflexion line, corresponding to incident and reflected waves, and no real roots after the reflexion line. The appropriate canonical form for the phase function is

$$\chi = \tfrac{1}{3}\xi^3 + \rho\xi + \sigma, \tag{B 4}$$

and the stationary phase condition becomes

$$\partial_\xi \chi = \xi^2 + \rho = 0. \tag{B 5}$$

This is called the 'fold' catastrophe and the canonical coordinate ρ demarcates the two sides of the caustic (i.e. the number of real roots). For the amplitude factors it suffices that there is linear dependence upon ξ:

$$\mathscr{A} = (1/2\pi)\,(A + \xi C), \quad \mathscr{B} = (1/2\pi)\,(B + \xi D). \tag{B 6}$$

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The reason for this is that if $\mathscr{A}$ or $\mathscr{B}$ had higher-order polynomial dependence upon ξ, then repeated integration by parts would reduce the polynomials, modulo $\partial_\xi \chi$, until a linear form was obtained. The inclusion of the factor $1/2\pi$ in equation (B 6) ensures that when the contour Γ is chosen to be along the real axis the ansatz (B 1) takes the more familiar Airy function form

$$u = \exp(\mathrm{i}\sigma(\boldsymbol{x}))\,\{[A(\boldsymbol{x},\boldsymbol{y})-\mathrm{i}B(\boldsymbol{x},\boldsymbol{y})]\,\mathrm{Ai}\,(\rho(\boldsymbol{x}))+\mathrm{i}[C(\boldsymbol{x},\boldsymbol{y})-\mathrm{i}D(\boldsymbol{x},\boldsymbol{y})]\,\mathrm{Ai}'\,(\rho(\boldsymbol{x}))\}. \quad \text{(B 7)}$$

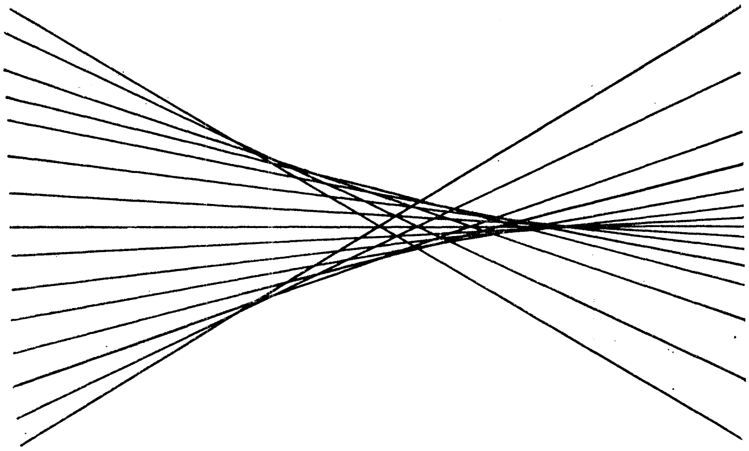

FIGURE 9. Ray paths near a cusped caustic.

For a cusped caustic there are three real roots between the caustic and one real root outside (see figure 9). The canonical form is that of the 'cusp' catastrophe

$$\chi = \tfrac{1}{4}\xi^4 - \tfrac{1}{2}\nu\xi^2 - \rho\xi + \sigma, \quad \text{(B 8)}$$

with the stationary phase condition

$$\partial_\xi \chi = \xi^3 - \nu\xi - \rho = 0. \quad \text{(B 9)}$$

In terms of the canonical coordinates ρ, ν the region between the caustics is given by

$$27\rho^2 < 4\nu^3. \quad \text{(B 10)}$$

Since $\partial_\xi \chi$ is a cubic, it suffices that the amplitude factors are quadratic in ξ:

$$\mathscr{A} = A + \xi C + \xi^2 E, \quad \mathscr{B} = B + \xi D + \xi^2 F. \quad \text{(B 11)}$$

Evaluating the integral (B 1) along the real axis gives an ansatz involving Pearcey functions (with the normalization and notation used by Hughes 1976):

$$u = \exp(\mathrm{i}\sigma)\,\{(A-\mathrm{i}B)\,I_0(\nu,\rho) + (C-\mathrm{i}D)\,I_1(\nu,\rho) + (E-\mathrm{i}F)\,I_2(\nu,\rho)\}. \quad \text{(B 12)}$$

The problem being addressed in this appendix is how the coefficients σ, ν, ρ, A, B, C, D, E, F involved in the uniformly-valid solution (B 12) can be determined from knowledge of the singular ray solutions

$$u \simeq \sum_{j=1}^{3} (a^{(j)} - \mathrm{i}b^{(j)}) \exp(\mathrm{i}\chi^{(j)}). \quad \text{(B 13)}$$

The first, and most difficult, aspect is to find σ, ν, ρ when given three values

$$\chi^{(j)} = \tfrac{1}{4}\xi_j^4 - \tfrac{1}{2}\nu\xi_j^2 - \rho\xi_j + \sigma,$$

where ξ_j are the three roots of the cubic equation

$$\xi^3 - \nu\xi - \rho = 0,$$

the $\chi^{(j)}$ being known from the solution (B 13).

Considerations of symmetry with respect to the three phases leads us to evaluate the linear, quadratic and cubic combinations

$$\bar{\chi} \equiv \tfrac{1}{3}\{\chi^{(1)} + \chi^{(2)} + \chi^{(3)}\} = \sigma - \tfrac{1}{6}\nu^2, \quad \text{(B 14)}$$

$$\Sigma \equiv -\tfrac{1}{3}\{(\chi^{(1)} - \bar{\chi})(\chi^{(2)} - \bar{\chi}) + (\chi^{(2)} - \bar{\chi})(\chi^{(3)} - \bar{\chi}) + (\chi^{(3)} - \bar{\chi})(\chi^{(1)} - \bar{\chi})\} = \tfrac{1}{144}\nu[\nu^3 + 54\rho^2], \quad \text{(B 15)}$$

$$\Pi \equiv (\chi^{(1)} - \bar{\chi})(\chi^{(2)} - \bar{\chi})(\chi^{(3)} - \bar{\chi}) = \tfrac{1}{864}\nu^6 - \tfrac{5}{32}\rho^2\nu^3 - \tfrac{27}{64}\rho^4. \quad \text{(B 16)}$$

The form of these results suggests that we should first solve equations (B 15, 16) for ρ, ν and subsequently solve equation (B 14) for σ. Note the care which must be taken in numerical work because of the differences of χs in Σ and Π.

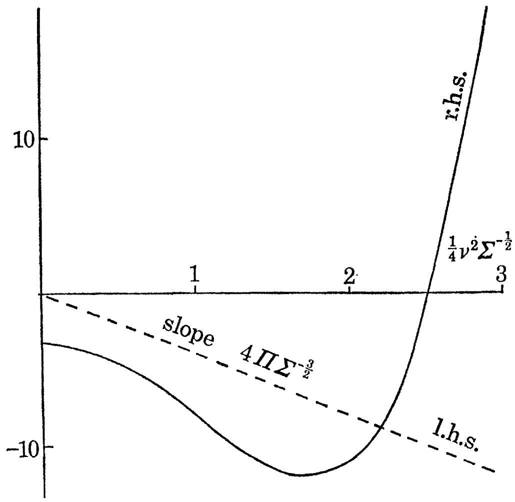

FIGURE 10. Graphical solution of the quartic (B 17) for ν^2.

We can eliminate ρ^2 to give a quartic for ν^2:

$$4\Pi(\tfrac{1}{4}\nu^2) = (\tfrac{1}{4}\nu^2)^4 - 6\Sigma(\tfrac{1}{4}\nu^2)^2 - 3\Sigma^2. \quad \text{(B 17)}$$

The graph (figure 10) of the two sides of this equation reveals that there is a unique positive root for ν^2, and hence for ν, provided that $-\Pi$ is sufficiently small. Using the explicit formula for the solution of quartic equations (Abramowitz & Stegun 1964, § 3.8) we obtain

$$\nu = 2\,\Sigma^{\frac{1}{4}}\{\operatorname{sgn}(\Pi)\,(1-\mu)^{\frac{1}{2}} + [2 + \mu + 2(1 + \mu + \mu^2)^{\frac{1}{2}}]^{\frac{1}{2}}\}^{\frac{1}{2}}, \quad \text{(B 18)}$$

with

$$\mu \equiv [1 - \tfrac{1}{4}\Pi^2\Sigma^{-3}]^{\frac{1}{3}} = \frac{(4\nu^3 - 27\rho^2)}{(\nu^3 + 54\rho^2)}\left[\frac{27\rho^2}{4\nu^3}\right]^{\frac{1}{3}}. \quad \text{(B 19)}$$

The condition (B 10) that there be three real rays ensures that μ is positive. It happens that this is precisely the bound that is needed to guarantee the uniqueness of the positive solution (B 18).

From equations (B 14, 15) it is now trivial to calculate the second canonical coordinate ρ and the phase function σ:

$$\rho = \pm[(144\Sigma - \nu^4)/54\nu]^{\frac{1}{2}}, \quad \sigma = \bar{\chi} + \tfrac{1}{6}\nu^2. \quad \text{(B 20, 21)}$$

Since ρ is a continuous function, the selection of sign can be resolved by arbitrarily choosing that ρ should be positive near one of the caustics and negative near the other.

Having determined the coordinates ρ and ν it is straightforward to determine the amplitude terms. First, we solve the cubic (B 9) to find the roots ξ_j associated with each of the three ray paths:

$$\xi_1 = r\cos\psi, \quad \xi_2 = -\tfrac{1}{2}r\cos\psi - \tfrac{1}{2}\sqrt{3}r\sin\psi, \quad \xi_3 = -\tfrac{1}{2}r\cos\psi + \tfrac{1}{2}\sqrt{3}r\sin\psi, \tag{B 22}$$

with

$$r^2 = \tfrac{4}{3}\nu, \quad \cos 3\psi = 27\rho^2/\nu^3. \tag{B 23}$$

Then we identify the amplitude factors in the alternative ray solutions (B 2) and (B 13):

$$\begin{bmatrix} 1 & \xi_1 & \xi_1^2 \\ 1 & \xi_2 & \xi_2^2 \\ 1 & \xi_3 & \xi_3^2 \end{bmatrix} \begin{bmatrix} A \\ C \\ E \end{bmatrix} = (2\pi)^{-\frac{1}{2}} \begin{bmatrix} (3\xi_1^2-\nu)^{\frac{1}{2}}\,a^{(1)} \\ (3\xi_2^2-\nu)^{\frac{1}{2}}\,a^{(2)} \\ (3\xi_3^2-\nu)^{\frac{1}{2}}\,a^{(3)} \end{bmatrix}, \tag{B 24}$$

with a similar equation for B, D, E in terms of $b^{(j)}$. These linear equations can be solved by means of Vandermonde determinants. For example, the explicit solution for A is

$$A = (2\pi)^{-\frac{1}{2}} \sum_{j=1}^{3} \frac{(3\xi_j^2-\nu)^{\frac{1}{2}}\,a^{(j)}\xi_{j+1}\xi_{j+2}}{(\xi_{j+1}-\xi_j)\,(\xi_{j+2}-\xi_j)}, \quad \text{with} \quad \xi_{j+3} = \xi_j. \tag{B 25}$$

The corresponding solutions for C and E can be obtained by replacing ξ_{j+1} and ξ_{j+2} in the numerator by ξ_j and 1 respectively.

To make practical use of the results derived above, the order of procedure would be as follows. First, the conventional ray solution would be found. This reveals the presence and position of caustics and cusps. Secondly, in the region between the cusped caustics the linear, quadratic and cubic combinations $\overline{\chi}$, Σ and Π of the phase functions would be evaluated. The solution for ν is then given explicitly by equations (B 18, 19) and ρ, σ follow from equations (B 20, 21). Thirdly, the Vandermonde equations (B 24) or (B 25) yield the amplitude factors. Finally, to construct the uniform solution (B 12) it is necessary to be able to evaluate the Pearcey functions. (If it is only the peak wave intensity that is of concern then it suffices to know that the global maximum of I_0 is about 3.6.) An important practical point is that all the parameters of the uniform solution vary smoothly. Thus numerical smoothing is permissable, and indeed desirable, in any implementation of the above procedure.

Appendix C. Higher-order dispersive effects upon near-linear waves near triple roots

The idea underlying the operator expansion method employed above in §§ 6 and 7, is that the essential character of waves near a ray coalescence can be revealed by a truncated series expansion of the dispersion relation. The retention of too few terms means that some features are lost, while the retention of too many terms gives unnecessary complexity. For caustics the appropriate level of truncation was obvious and leads to a model equation

$$\mathrm{i}G_\omega \frac{\partial a}{\partial t} - \tfrac{1}{2}G_{ll}\frac{\partial^2 a}{\partial x^2} + xG_x a + H|a|^2 a = 0, \tag{C 1}$$

in which G_ω, etc., are evaluated at $(\omega_0, \boldsymbol{k}_0, 0)$. At a triple root the coefficient G_{ll} is zero. Thus, one might expect that it would suffice simply to retain the G_{lll} term:

$$\mathrm{i}G_\omega \frac{\partial a}{\partial t} + \tfrac{1}{6}\mathrm{i}G_{lll}\frac{\partial^3 a}{\partial x^3} + xG_x a + H|a|^2 a = 0. \tag{C 2}$$

While this is an adequate model of the waves when the triple-root condition is met exactly, it fails to describe the change of behaviour when the wave frequency is slightly detuned.

One means of resolving this problem is to use formal asymptotic approximations involving a small parameter. For example, the work of Smith (1976*b*) reveals that there are rapid variations on three distinct length scales ϵ, $\epsilon^{\frac{3}{4}}$, $\epsilon^{\frac{1}{2}}$. An interpretation of this behaviour is that the appropriate level of truncation is different in three different directions in the parameter space (ω, l, x). The orientations of these directions are related to the topology of the linear dispersion relation (see figure 11). This leads us to make a heuristic derivation based upon the use of catastrophe theory.

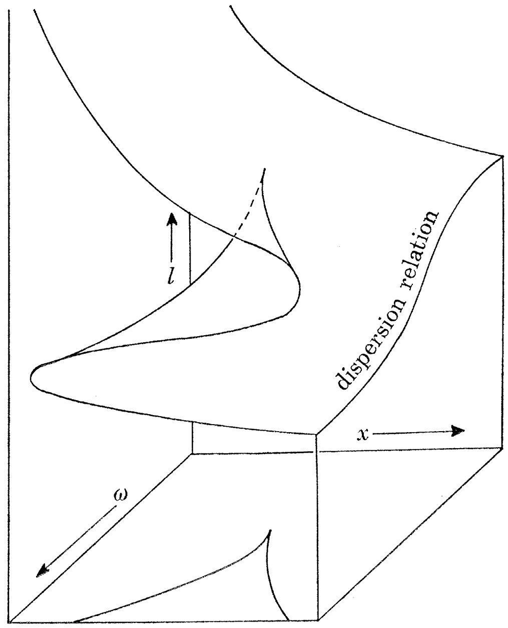

FIGURE 11. The linear dispersion equation as a folded surface showing orientation of axes.

The projection of the dispersion relation onto the (ω, x) plane is a cusp. There are three roots for l inside the cusp, and outside there is only one root. Not surprisingly, this is precisely the type of root behaviour that we have been studying in the previous appendix. Thus, the linear dispersion relation has the canonical representation

$$(l-l_0)^3 - \tilde{\nu}(l-l_0) - \tilde{\rho} = 0, \qquad \text{(C 3)}$$

where $\tilde{\nu}$, $\tilde{\rho}$ are functions of x and ω. Moreover, any local representation must have terms corresponding to both $\tilde{\nu}$ and $\tilde{\rho}$ if it is to preserve the essential character of the root coalescence. The inadequacy of equation (C 2) can be attributed to the absence of a counterpart to $\tilde{\nu}(l-l_0)$. That is, the absence of a term involving a single derivative with respect to x.

The canonical quantities $\tilde{\rho}$ and $\tilde{\nu}$ respectively can be thought of as being generalisations of the distance from the coalescence point and the frequency detuning. Far from the triple-root position the detuned waves can be expected to resemble more nearly an exact triple root. Thus, in seeking to improve upon the model equation (C 2) it is necessary to evaluate the $\tilde{\nu}$ term at $\tilde{\rho} = 0$. This simple observation is in fact the crux of a successful heuristic derivation. The failure of equation (C 2) can be attributed to an incorrect choice of the curve in (ω, x) space along which the $\tilde{\nu}$ term is evaluated. Catastrophe theory helps to define the important orientations.

By taking the Taylor series expansion of $G(\omega, l, x)$, substituting for $(l-l_0)^3$ from the cubic (C 3) and equating powers of $(l-l_0)$, we find that near the triple point

$$-\tfrac{1}{6}G_{lll}\tilde{\rho} = (\omega-\omega_0)\, G_\omega + xG_x + \dots,$$
$$-\tfrac{1}{6}G_{lll}\tilde{\nu} = (\omega-\omega_0)\, G_{\omega l} + xG_{xl} + \dots, \quad \text{(C 4)}$$

where all the partial derivatives are evaluated at the triple point. In particular, along the curve $\tilde{\rho} = 0$ we have $x = -(\omega-\omega_0)\, G_\omega/G_x$, and thus

$$-\tfrac{1}{6}G_{lll}\tilde{\nu} = (\omega-\omega_0)\,[G_{xl} - G_\omega G_{xl}/G_x]. \quad \text{(C 5)}$$

Hence, correct to leading order in $(\omega-\omega_0)$ and x, the canonical representation (C 3) of the linear dispersion relation becomes

$$\tfrac{1}{6}G_{lll}(l-l_0)^3 + [G_{\omega l} - G_\omega G_{xl}/G_x]\,(\omega-\omega_0)\,(l-l_0) + (\omega-\omega_0)\, G_\omega + xG_x = 0. \quad \text{(C 6)}$$

If now we include the local first approximation to the nonlinear term and we replace $-\mathrm{i}(\omega-\omega_0)$, $\mathrm{i}(l-l_0)$ by partial derivatives, we find that the resulting model equation is

$$\mathrm{i}G_\omega \frac{\partial a}{\partial t} + [G_{\omega l} - G_\omega G_{xl}/G_x]\,\frac{\partial^2 a}{\partial x\,\partial t} + \tfrac{1}{6}\mathrm{i}G_{lll}\,\frac{\partial^3 a}{\partial x^3} + xG_x a + H|a|^2 a = 0. \quad \text{(C 7)}$$

As a partial check upon the validity of equation (C 7) we can verify that for infinitesimal, strictly periodic waves the solution is a Pearcey function, and furthermore, the arguments agree with the first two terms of a formal perturbation expansion derived by Smith (1976). Specifically, if we use the local amplitude A_0 of the linear h.o.d. solution then for periodic waves equation (C 7) can be transformed to the standard form

$$-\mathrm{i}\,\frac{\mathrm{d}^3 I}{\mathrm{d}\rho^3} - \mathrm{i}\nu_0\,\frac{\mathrm{d}I}{\mathrm{d}\rho} - \rho I + \beta|I|^2 I = 0,$$

with the far field conditions

$$I \simeq |\rho|^{\frac{1}{3}}\,(\tfrac{2}{3}\pi)^{\frac{1}{2}} \exp\{-\mathrm{i}[\tfrac{3}{4}\rho^{\frac{4}{3}} + \tfrac{1}{2}\nu_0\rho^{\frac{2}{3}} \pm \text{const.}]\} \quad \text{as} \quad \rho \to \pm\infty.$$

For G_x/G_{lll} positive the details of the transformation are

$$a = A_0 I(\nu_0, \rho; \beta), \quad \beta = -A_0^2 H|\tfrac{1}{6}G_{lll}G_x^3|^{-\frac{1}{4}},$$
$$\nu_0 = -(\omega-\omega_0)\,[G_x G_{\omega l} - G_\omega G_{xl}]\,|\tfrac{1}{6}G_{lll}G_x^3|^{-\frac{1}{2}},$$
$$\rho = \{(\omega-\omega_0)\, G_\omega + xG_x\}\,|\tfrac{1}{6}G_{lll}G_x^3|^{-\frac{1}{4}},$$

the expressions for the arguments ν_0, ρ, β of the Pearcey function agreeing with Smith's equations (13*a*, *c*, *d*). For G_x/G_{lll} negative it suffices to replace I by its complex conjugate and to reverse the sign of β and ρ in the above equations.

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